The PQ-duality between the trigonometric Calogero model and the rational Ruijsenaars model as a spectral duality

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Abstract

Two well-known examples of dualities between integrable systems are the spectral duality and the Ruijsenaars (or PQ-) duality. We have shown that PQ-duality between rational Calogero system and itself, or between trigonometric Calogero system and Heisenberg chain, can be described in terms of spectral duality. In particular, one can make a gauge transformation that adds dependence on the spectral parameter to the Lax matrix without changing the Lax equation. The spectral dual system after the inverse gauge transformation of the Lax matrix turns into the Lax matrix of the PQ-dual model.

1. PQ- or Ruijsenaars duality

The Ruijsenaars duality for N-particle systems builds a correspondence between the coordinate variables of one system and the action variables of another: $(q_i, I_i) \leftrightarrow (\tilde{I}_i, \tilde{q}_i)$. Such a duality is known, for example, between the rational Calogero model and itself and between the trigonometric Calogero model and the rational Ruijsenaars-Schneider model.

The phase space of the rational Calogero model is obtained by the Hamiltonian reduction of $\mathrm{T}^*\mathrm{gl}_N = \mathrm{gl}_N * \mathrm{gl}_N$. Performing a Hamiltonian reduction, we fix a momentum map level:

$$\mu(A,B) = [A,B] = \nu\overline{O}, \quad \overline{O}_{ij} = 1-\delta_{ij}, \quad A,B \in \mathrm{gl}_N$$

Diagonalizing different matrices while resolving momentum equation, we obtain either $[Q,L] = \nu\overline{O}$ equation or $[\tilde{L},\tilde{Q}] = \nu\overline{O}$, where L is the Lax matrix of Calogero system:

$$(L)_{ij} = \delta_{ij}p_i + \nu\frac{(1-\delta_{ij})}{q_i-q_j}, \quad i,j \in \overline{1,N}.$$

and $\tilde{L}$ is the Lax matrix of the dual one. Both systems are connected by adjoint action: $\mathrm{Ad}_{D\Psi^{-1}}$, $D = \delta_{ij}\sum_k \Psi_{ki}$.

In case of trigonometric Calogero model, the phase space is $\mathrm{T}^*\mathrm{Gl}_N = \mathrm{Gl}_N * \mathrm{gl}_N$, and the moment equation is:

$$\mu(A,B) = A - BAB^{-1} = \nu\overline{O}, \quad \overline{O}_{ij} = 1-\delta_{ij}, \quad A \in \mathrm{gl}_N,\ B \in \mathrm{Gl}_N$$

Resolving this equation with respect to matrices A or B, we obtain either trigonometric Calogero model:

$$(L)_{ij} = \delta_{ij}p_i - \nu(1-\delta_{ij})\frac{e^{-q_{ij}/2}}{\mathrm{sh}(q_{ij}/2)}, \quad q_{ij} = q_i - q_j, \tag{1}$$

or rational Ruijsenaars-Schneider system:

$$(\tilde{L})_{ij} = \frac{\nu}{\tilde{q}_i - \tilde{q}_j + \nu} e^{\tilde{p}_j} \prod_{k\neq j} \frac{\tilde{q}_j - \tilde{q}_k - \nu}{\tilde{q}_j - \tilde{q}_k}.$$

2. Spectral duality

Spectral duality describes the duality between two spin systems that have a Lax representation that depends on the spectral parameter z. The spectral curves for such dual systems coincide:

$$\Gamma(\lambda, z): \quad \det(L(z) - \lambda) = 0$$

A basic example of such a duality is rational Gaudin system:

$$L(z) = \Lambda + \sum_{a=1}^{N} \frac{S^a}{z - z_a}, \quad (\Lambda)_{ij} = \delta_{ij}\lambda_i, \quad S, \Lambda \in \mathrm{Mat}(M\times M). \tag{2}$$

In special reduced case of $\mathrm{rk}\, S^a = 1$, we can write spin variables as: $(S)^a_{ij} = \xi_{ia}\eta_{aj}$. For new variables we have canonical Poisson bracket:

$$\{S^a_{ij}, S^b_{kl}\} = \delta_{ab}(\delta_{kj}S^a_{il} - \delta_{il}S^a_{kj}) \ \rightarrow\ \{\eta_{ia}, \xi_{jb}\} = \delta_{ij}\delta_{ab}$$

If we interchange marked points z_i and parameters λ_i and consider Gaudin system with a slightly different spin variables $A^i = \eta_{ia}\xi_{aj}$:

$$L'(z) = Z + \sum_{i=1}^{M} \frac{A^i}{z - \lambda_i}, \quad (Z)_{ij} = \delta_{ij}z_i, \quad A, Z \in \mathrm{Mat}(N\times N), \tag{3}$$

we obtain that spectral curves of models (2) and (3) coincide:

$$\det(L(z) - \lambda) = \mathrm{const}\cdot\det(L'(\lambda) - z).$$

Another example [1] of spectral dual systems are the trigonometric Gaudin model:

$$L(z) = \sum_a \mathrm{cth}(z - z_a)E_{ii}S^a_{ii} + \sum_a \frac{1}{\mathrm{sh}(z-z_a)} \sum_{i<j} (e^{z-z_a}E_{ij}S^a_{ij} + e^{-z+z_a}E_{ji}S^a_{ji})$$

and the Heisenberg chain:

$$T(x) = \prod_j (x - x_j)V(q)\left(1 + \sum_i \frac{A^i}{x - x_i}\right).$$

3. Rational case

It turns out that the PQ-duality can be described from the point of view of spectral duality. Indeed, consider the Lax matrix for the rational Calogero-Moser model (1). We can make a gauge transformation introducing the spectral parameter z:

$$L \rightarrow L'(z) = gL^Cg^{-1}, \quad (g)_{ij} = \delta_{ij}(z - q_i),$$

then we get:

$$(L'(z))_{ij} = \left(\delta_{ij}p_i + \nu\frac{(1-\delta_{ij})}{q_i - q_j}\right) - (1-\delta_{ij})\frac{\nu}{z - q_j}.$$

The resulting Lax matrix can be represented as a sum of rational Calogero-Moser Lax matrix and a degenerate Gaudin-type Lax matrix:

$$(L'(z))_{ij} = L^{Cal} - \nu\sum_{a=1}^{N}\frac{\overline{O}^a}{z - q_a} = L^{Cal} - L^{Gaud}(z), \quad (\overline{O}^a)_{ij} = (1-\delta_{ij})\delta_{aj}.$$

Making gauge transformation by eigenvector matrix Ψ ($L\Psi = \Psi\Lambda$), we obtain:

$$\tilde{L}(z) = \Psi^{-1}L'(z)\Psi = \Lambda - \nu\sum_{a=1}^{N}\frac{S^a}{z - q_a},$$

where Λ is diagonal matrix of action variables and new spin variable ($\mathrm{rk}\, S^i = 1$), and can be represented as $S^a = \xi_{ia}\eta_{aj}$, where $\eta_{aj} = \Psi_{aj}$ and $\xi_{ia} = \sum_k \Psi^{-1}_{ik}(1-\delta_{ka})$. If we consider the spectral dual Gaudin system (3):

$$\tilde{L}(\lambda) = Q - \nu\sum_{a=1}^{N}\frac{\eta_{ia}\xi_{aj}}{z - \lambda_a},$$

and again perform a gauge transformation:

$$\tilde{L}(\lambda) \rightarrow \tilde{L} = gD\Psi^{-1}\tilde{L}(\lambda)\Psi D^{-1}g^{-1}, \quad D = \delta_{ij}\sum_k \Psi_{ki},$$

we will obtain a PQ-dual Calogero-Moser system.

4. Trigonometric case (work in progress)

The same idea also works in trigonometric case. If we consider trigonometric Calogero model (1) and perform gauge transformation:

$$L \rightarrow L'(z) = gL^Cg^{-1}, \quad (g)_{ij} = \delta_{ij}\frac{e^{\frac{q_i}{2}}}{\mathrm{sh}(\frac{z-q_i}{2})},$$

we again will obtain the Lax matrix which is represented as a sum of initial Calogero matrix and a Gaudin like term:

$$L'(z) = L_{\mathrm{Cal}} - \nu\sum_a \frac{e^{\frac{z-q_a}{2}}\overline{O}^a}{2\,\mathrm{sh}(\frac{z-q_a}{2})}, \quad (\overline{O}^a)_{ij} = (1-\delta_{ij})\delta_{aj}.$$

We expect that after some gauge transformation we will obtain a system dual to some form of Heisenberg chain, which after inverse gauge transformation will become rational Ruijsenaars model.

References

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